

Implementation of the Frequency Distribution Approach in Pricing Maximum Options

A Master Thesis

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Declaration of Authorship

I hereby confirm that I have authored this thesis independently and without use of other than the indicated resources.

All passages, which are literally or in general manner taken out of publications or other sources are marked as such.

Berlin, Germany

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March 20th, 2002

Acknowledgement

At the time I sat for my first ever lesson as a student at the Humboldt Universität zu Berlin, I really just had a vague idea of what options were all about, how the major stock traders and brokers got along with the value of derivative securities at the Stock Exchange markets was a puzzling issue I still had to come to terms with.

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I hope the contents of this thesis do serve a contributory purpose to the field of financial mathematics.

Berlin, Germany

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Abstract

This thesis briefly reviews the use of the Black–Scholes pricing model, Finite difference methods, binomial pricing model and Monte Carlo simulations in pricing path-dependent options.

The main objective of this thesis however, is the implementation of the frequency distribution method as an analytic and more accurate approach to pricing such options. This approach revolves around evaluating the distribution of stock prices within a binomial lattice structure. A discrete time re-combining binomial model is used to develop closed-form analytic expressions for the frequency distribution of path maxima over any finite horizon. The frequency distribution approach provides an immediate and direct solution to the value of the European option. Following this path-maxima approach, if the interest rates are zero, the values of the American and European options will be equal. But in case the interest rates are positive, the value of the American option will be higher than that of a corresponding European option. Our procedure in this case is to derive a Markovian relation among bundles of paths in the frequency distribution matrix through which we can efficiently and accurately value the American option.

In contrast to the other pricing procedures which will be explored in the course of the literature survey, the methods proposed here have the basic advantage of easing computational difficulties, while increasing efficiency, speed and accuracy in path-dependent option pricing. Moreover, these methods can also be extended to value all other forms of path-dependent options that were mostly priced through continuous time approaches. The analytical approaches are supplemented by a customized computer program on typical pricing examples of American and European options. This sample programming algorithm which can be run on an **XploRe** statistical software is included.

Chapter 1

Introduction

1.1 Overview

In current financial markets, accurate pricing of the ever increasing volumes of sophisticated derivative securities, has remained an active area of research. Most up-to-date pricing algorithms based on the Black-Scholes pricing model, the binomial pricing approach, the finite difference as well as the Monte Carlo simulation methods, yield fairly accurate results for European, American and Asian options, but are often time-consuming, computationally complex, and plagued by convergence difficulties. In line with the ongoing research, this thesis presents yet another option pricing approach with the objective of improving pricing accuracy while maintaining computational efficiency and speed.

A **European** call (respectively put) option is a contract in which the owner may (without obligation) buy (respectively sell) some prescribed asset (the underlying) S , at a prescribed time (expiry date) T , and at a fixed price (exercise or strike price) E . In contrast to a European option, the **American** style option can be exercised at any time during the lifetime of the option, and not just at expiry. This property has been a major source of difficulty in the valuation of American options.

Many valuation codes make use of the continuous-time Black-Scholes pricing model which assumes that the asset price is driven by a Brownian motion, and specifies a stochastic differential

equation that the option must satisfy (see Black & Scholes, 1973). This valuation process achieved by solving the partial differential equations(PDE) provides explicit closed form solutions for the values of certain (European style) call and put options. However for complex Asian and American lookback (path-dependent) options, the Black-Scholes differential equation either has no closed form solutions, or if such closed form solutions exist, the formulas expressing them are complicated and difficult to evaluate accurately by conventional methods. This valuation method will be re-examined in the next chapter of this thesis.

Another widely applied method is with the use of the binomial pricing model from Cox, Ross & Rubinstein (1979), based on a simpler discrete-time process. The mathematical justification behind this model is that the standard symmetric random walk, when appropriately scaled converges to a Brownian motion (see section 2.1). However, the straightforward application of this method to value path-dependent options is particularly difficult. This is because the future cash flows depend on the entire stock price process rather than on just the final stock price, whereupon there exist 2^n possible paths to this price.

In this thesis we present analytical procedures for developing frequency distribution approaches to valuing European and American maximum options on a discrete time , re-combining binomial process based on the methods from Neave & Slavinsky (2001) . The first part of this thesis is dedicated to a review of the notion of path-dependency, and some of the popularly traded examples of options. The same chapter will treat the current approaches that have been used to value path-dependent options and the setbacks associated with the use of these methods.

The subsequent chapters deal with the main theme of the thesis. These sections elaborate the use of analytical expressions for developing the frequency distribution $A(T)$ that describes the number of paths taken by the option. A set of paths ending at time T , with price j and attaining the maximum m at some time $t = 0, 1, 2, \dots, T$ will henceforth be referred to as a *bundle*. Since the frequency distribution is obtained as a closed form solution, the value of a European option is simply calculated by evaluating the payoff in each set of paths. The exact value of an American option can then be calculated by deriving an efficient Markovian relation among the bundles, that uses just the required number of state variables (Neave (1997), Neave & Stein (1998) and Neave & Ye (1999)). This method of combining the frequency distribution with the recursive markovian relation

accurately prices European options with computational complexity of T^3 , and T^4 for the American options, where T represents any time horizon. For the purpose of these study, examples will be focused on calls on the maxima, however as will be mentioned later in this thesis, the methods can be extended to value all other forms of path-dependent options.

1.2 Literature survey

In this section we will briefly review the theory of path-dependent options, the related modern concepts of continuous and discrete time models as well as their pricing methods.

1.2.1 Path-dependency

An option is an example of a derivative security, so named because its value is derived from that of the underlying asset. By arranging now to buy a commodity in the future, the client ensures that the the price at which he can trade is fixed, this is the case of a *forward*, the simplest example of a financial derivative. A standardized form of this forward contract which is traded by Hedgers, Speculators and Arbitrageurs in financial markets is referred to as a *future*. Path-dependent options are options whose payoffs depend on the historical values of the underlying asset over a given time period as well as its current value. Over the last ten years these options have become increasingly popular in equity, commodity and Foreign Exchange markets. These options are defined using either discrete or continuous price sampling, well known variants include look-back, Asian, barrier and capped options.

An ordinary **Call (or Put)** option gives its owner the right to buy (sell) the underlying(S) for the strike price (K) on the expiration date (T), expressed mathematically as:

$$C_T = \max(0, S_T - K)$$

$$P_T = \max(0, K - S_T)$$

On the expiry date, if an option is traded on a positive C_T (or P_T), the option is considered to be in the money, else it is out of the money. The option is otherwise considered to be on the money if the value of the underlying asset (S_T) is equal to that of the strike price (K). In the same context,

according to Franke, Härdle & Hafner (2001), the value of a call on an American or European option as a function of the strike price K is monotone decreasing. This is because the value of a call increases with a decrease in the strike price.

$$C_{K_1,T}(s, \tau) \geq C_{K_2,T}(s, \tau)$$

On the other hand, that of a Put as a function of the strike price is monotone increasing.

$$P_{K_1,T}(s, \tau) \leq P_{K_2,T}(s, \tau)$$

where $\tau = T - t$ is the time to expiry of the option, and $K_1 \leq K_2$.

Let the maximum (M_t) and minimum (m_t) values of an asset S for an arbitrary date t_1 over the time period $[0, T]$ be

$$\begin{aligned} M_{t_1} &= \max\{S_t : t_1 \leq t \leq T\} \\ m_{t_1} &= \min\{S_t : t_1 \leq t \leq T\} \end{aligned} \tag{1.1}$$

where T is the expiry date, and $t = 0$ denotes the present ($0 \leq t_1 \leq T$).

We can then define a **look-back** call (respectively put) option as an option whose payoff depends not only on the asset price at expiry, but also on the maximum (resp. minimum) value of the underlying asset over the time period $[0, T]$

$$C_T = S_T - m_T$$

$$P_T = M_T - S_T$$

A **maximum** (respectively minimum) call option (option on the extrema) is similar to an ordinary call (put) option, however the price of the underlying asset depends on the historical maximum (or minimum) value attained by the option over the time period $[0, T]$

$$C_T = \max(0, M_T - K)$$

$$P_T = \min(0, K - m_T)$$

A **capped** call (or put) option remains an ordinary option as long as the historical maximum (or minimum) of the underlying asset lies below a predefined upper (resp. lower) barrier price b . If the maximum (or minimum) attains that barrier, the option is automatically exercised.

$$C_T = \begin{cases} b - K & \text{if } M_T \leq b \\ \max(0, S_T - K) & \text{otherwise} \end{cases}$$

$$P_T = \begin{cases} K - b & \text{if } m_T \geq b \\ \max(0, K - S_T) & \text{otherwise} \end{cases} \quad (1.2)$$

Barrier options can be examined under two minor categories; *Down and Out (or Up and Out)* call (respectively put) barrier options are those whose payoff is the same as that of an ordinary option as long as the historical minimum (resp. maximum) of the underlying remains above (or below) a predefined upper (or lower) barrier price b , else the payoff is 0.

$$C_T = \begin{cases} \max(0, S_T - K) & \text{if } m_T > b \\ 0 & \text{otherwise} \end{cases} \quad (1.3)$$

$$P_T = \begin{cases} \max(0, K - S_T) & \text{if } M_T < b \\ 0 & \text{otherwise} \end{cases} \quad (1.4)$$

Down and In (or Up and In) call (resp. put) barrier options are treated as ordinary options as long as their historical maximum (resp. minimum) stays below (or above) a predefined barrier b , else the option payoff is 0

$$C_T = \begin{cases} \max(0, S_T - K) & \text{if } M_T < b \\ 0 & \text{otherwise} \end{cases} \quad (1.5)$$

$$P_T = \begin{cases} \max(0, K - S_T) & \text{if } m_T > b \\ 0 & \text{otherwise} \end{cases} \quad (1.6)$$

Asian or **Average-rate** options are options whose payoff depends on the arithmetic or geometric average of the historical prices of the options over the time period $[0, T]$. Two basic types of options can be distinguished in this class; The *fixed-strike* average-rate options in which the value of the underlying asset S_T is substituted by the time average A_t ;

$$C_T = \max(0, A_T - K)$$

$$P_T = \max(0, K - A_T)$$

where

$$A_t = \frac{1}{t} \int_0^t S_\tau d\tau \quad (1.7)$$

In the *floating-strike* average-rate option, the strike price is substituted by the time average A_t

$$C_T = \max(0, S_T - A_t)$$

$$P_T = \max(0, A_t - S_T)$$

Determining a fair market value of any option has always been a problem in option pricing, since the behaviour of a financial security is subject to a series of changes in the market situation. Though some of these factors may be foreseen, the dimension of their effects remains difficult to predict. A path-dependent option is particularly difficult to value, since the future cash flow depends on the entire price path of the option rather than on just the final stock price.

Chapter 2

Option Pricing Models

The last two decades have been marked by the appearance of different types of options in financial markets, and because they are all designed to cover individual circumstances, analytic methods to price these derivatives have not always been available when the derivative securities are developed. However, fair estimates of the value of options can be obtained through the concepts of *risk-neutral pricing* or arbitrage.

2.1 The pricing theory

The first step in derivative pricing consists of making assumptions about the mathematical process governing the underlying stochastic variable in the model under consideration. *Risk neutrality* is the characteristic ascribed to an investor who is indifferent with respect to risk (Merton, 1971). This risk-neutral pricing principle which was pioneered by Cox and Ross (1976) indicates that any deflated asset price should be a martingale. Arbitrage opportunities cannot last for long in an efficient market, this is because the combination of the profit intentions and nearly instantaneous trading will almost always lead to a convergence of prices in the markets. The most important use of the notion of risk neutrality to us is its application to risk neutral valuation problems which will be reviewed in the next section.

The price of an option is typically a non-linear function of the underlying asset (and additional variables such as interest rates, strike, etc.). In actual fact, the behaviour of a normal stock option

in financial markets is known to follow a *stochastic process*, since its value changes over time in an uncertain way. As a consequence of this stochastic property, option prices can be evaluated from discrete time as well as continuous time perspectives. An example of a stochastic process that is popularly used in finance cycles is the *Markov process*.

A Markov relation is a particular type of stochastic process where only the present value of the variable is relevant for predicting the future. The past history of the variable and the way in which the present has emerged from the past is irrelevant. An example of a Markov process is the well known Brownian motion also referred to as the *Wiener Process*.

2.1.1 Brownian motion

The Brownian motion can be constructed by considering a simple symmetric random walk from an example involving a sequence of repeated tosses of a fair coin. We let the random variable X_i denote the equally likely occurrence of a Head or Tail after each toss of the coin;

$$X_i = \begin{cases} +1 & \text{if } H, \text{ the toss yields Head } k \\ -1 & \text{otherwise} \end{cases} \quad (2.1)$$

From the original starting point of 0, the result after n tosses will be given as $Y_n = \sum_{i=1}^n X_i$. This random process $\{Y_n, n = 0, 2, \dots, N\}$ describes the entire coin tossing process over time. Therefore to carry out the N -steps in time T , each step takes $\Delta t = T/N$ time units. Since the occurrences of Heads or Tails are independent events such that $E[X] = 0$ and $Var[X] = 1$, if we let $N \rightarrow \infty$,

$$E[Y_n] = 0, \quad var[Y_n] = n$$

Following the central limit theorem, if N is large enough, $Y_N/\sqrt{N}$ has approximately a normal distribution with mean 0 and variance 1. By implication our space steps are scaled as the square root of time.

As a consequence of this theorem, when $N \rightarrow \infty$, the sequence of random processes

$$Y^{(N)} = \{Y_{[tN]}/\sqrt{N}, \quad t \in [0, T]\} \text{ converges to a Brownian motion.}$$

2.1.2 The Wiener Process

This is a special type of Markov stochastic process which describes the behaviour of a variable $\{X_t, t \in [0, T]\}$ with the property that

$$\Delta X \sim N(0, \sqrt{\Delta t}),$$

where ΔX represents the change in the variable X in time Δt , and $N(\mu, \sigma)$ is the normal distribution function with mean μ and variance σ .

1. X_t is a continuous function of time, and $X_0 = 0$.
2. for each $t > 0$ and $s > 0$, $X_{t+s} - X_t$ is also normally distributed with mean 0 and variance s .
3. $X_{t+s} - X_t$ is independent of X_t .

We should note here that a Wiener process is also a continuous process with any collection of times having a multivariate normal distribution with mean 0 and covariance given as $E[X_t X_s] = \min(t, s)$, though however the existence of such a process is not trivial.

2.2 Analytic Pricing Procedures

Using the assumptions of risk neutrality, Duffie (1996) and Hull (1999) have shown that the price of a derivative security can be found as the expected value of its discounted payoffs when the expected value is taken with respect to a transformation of the original probability measure referred to as the risk-neutral measure.

Given the price of an underlying asset S_t , the strike price K , the time to expiry T , a risk-free interest rate r , Black & Scholes (1973) have shown that the behaviour of S_t as a function of time t is governed by a stochastic process generating the price of a non-dividend-paying stock in the form of a geometric Brownian motion;

$$dS_t = \mu S_t dt + \sigma S_t dX_t \tag{2.2}$$

where dX_t represents a standardized Wiener process on a suitable probability space $(\Omega, \mathcal{F}, P)$, μ (a measure of the average rate of growth of the asset price) is the drift rate, that is, the instantaneous

expected return of the underlying asset $S = (S_t)$ where $t > 0$ and σ is the volatility of the assets, which characterizes the fluctuations in the price S (see Franke, Härdle and Hafner, 2001).

2.2.1 The black–Scholes Pricing Model

By assuming the ideal financial market situation, models are developed where the underlying asset is characterized by a stochastic lognormal diffusion process, in this framework, the resulting portfolio is constructed continuously. The Black–Scholes model that is discussed here is based on the Arbitrage Pricing Theory of Duffie (1996) and Hull (1999), it assumes that trading at financial markets proceeds continuously in time, and that the asset price is driven by a Brownian motion specifying a stochastic differential equation which the option must follow.

For the value V of a European call or put option, having strike price K and expiry date T , Black and Scholes constructed a hedge portfolio containing solely the underlying asset S and a riskless rate r , which replicates the value of the option at each instant t in time. This equilibrium condition resulted in the diffusion-type, Black–Scholes partial differential equation (PDE) for the unknown function V :

$$\frac{\partial V}{\partial t}(t, S_t) + rS_t \frac{\partial V}{\partial S}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t, S_t) - rV(t, S_t) = 0$$

or simply written as

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} = rV \quad (2.3)$$

Since the option value V depends on time t and the asset price S , we henceforth use the notation $V(t, S)$. At expiry, the price of the option must equal the payoff, therefore the final condition to equation (2.3) for a European call option is

$$V(S, T) = \max(0, S - K)$$

with boundary conditions ($f : \mapsto \{x - K\}^+$) as the terminal payoff function

$$V(t, 0) = 0, \quad \text{and} \quad V(t, S) \sim S \quad \text{as} \quad S \rightarrow \infty$$

In the same context, the final condition for a European put option in equation (2.3) is given as

$$V(S, T) = \max(K - S, 0)$$

with boundary conditions

$$V(t, 0) = E[e^{-r(T-t)}], \quad \text{and} \quad V(t, S) \sim 0 \quad \text{as} \quad S \rightarrow \infty$$

We note here that when expressed in terms of the asset price, the B-S equation in (2.3) is independent of the drift rate μ , that is, the option value does not depend on investors' risk preferences. The hedging portfolio eliminates all the risk, hence there should be no extra reward for holding the option. The B-S PDE has a probabilistic solution of the form

$$V(t, x) = e^{-r(T-t)} E_q(\Psi(S_T) | S_t = x).$$

where q is the risk neutral probability and $\Psi(S_T) = V(S, T)$ is the final condition on the call or put option. This asset price process must satisfy the risk neutral stochastic differential equation

$$\frac{dS_t}{S_t} = rdt + \sigma dX_t$$

whose solution is

$$S_T = S_0 e^{\{(r - \frac{1}{2}\sigma^2)T + \sigma X_T\}} \quad (2.4)$$

Explicit closed form solutions exist for both cases, namely for the European call option;

$$V(t, S) = C(t, S) = e^{-r(T-t)} E[\max(S_T - K, 0) | F_t] \quad (2.5)$$

In terms of the cumulative distribution function of the normal distribution $\Phi = \mathcal{N}(\cdot)$, equation (2.5) will henceforth be expressed as

$$V(t, S) = C(t, S) = S \mathcal{N}(d_1) - K e^{-r(T-t)} \mathcal{N}(d_2) \quad (2.6)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S/K) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\mathcal{N}(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-\frac{x^2}{2}} dx$$

where $\mathcal{N}(\cdot)$ is the cumulative standard normal distribution. Using the same expressions for d_1 , d_2 and $\mathcal{N}(\cdot)$ above, the European put option is then expressed as

$$V(t, S) = P(t, S) = K e^{-r(T-t)} \mathcal{N}(-d_2) - S \mathcal{N}(-d_1) \quad (2.7)$$

or with a quicker approach through the put-call parity equation

$$C_t - P_t = S_t - Ke^{-r(T-t)}$$

The use of the Black–Scholes method provides good solutions for European put and call options, however for most other style options (especially the American and path-dependent alternatives), this model provides no closed form solutions. In recent research in financial mathematics, approximate analytical methods and numerical approaches have been developed to handle these circumstances.

2.2.2 The Barone–Adesi and Whaley Analytic approximation

In their quadratic approximation approach, Barone-Adesi & Whaley (1987) suggested approximating the difference v , between European and American option prices in the differential equation

$$\frac{\partial v}{\partial t} + (r - q)S \frac{\partial v}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} = rv \quad (2.8)$$

where q represents the rate at which the stock is paying on a continuous dividend yield.

Applying appropriate limiting boundary conditions to equation (2.8), the value of an American Call $C(S_A, T)$ would be given as

$$C(S_A, T) = \begin{cases} C(S_E, T) + A_2\left(\frac{S}{S^*}\right)^{\gamma_2} & \text{when } S < S^* \\ S - X & \text{when } S \geq S^* \end{cases} \quad (2.9)$$

Where S^* is the capped price of the option (above which the option must be exercised). The estimate of S^* iteratively solves the equation

$$g(S^*) = S^* - X - c(S^*) - \left\{ 1 - e^{q(T-t)} N[d_1(S^*)] \right\} \frac{S^*}{\gamma_2}. \quad (2.10)$$

where

$$\begin{aligned} M &= \frac{2r}{\sigma^2}, \\ N &= \frac{2(r - q)}{\sigma^2}, \\ K(\tau) &= 1 - e^{-r(\tau)}, \quad \text{and} \quad \tau = T - t \end{aligned}$$

$$\gamma_2 = \frac{1}{2} \left[-(N-1) + \sqrt{(N-1)^2 + \frac{4M}{K}} \right] \quad (2.11)$$

$$A_2 = \left(\frac{S^*}{\gamma_2} \right) \{1 - e^{-q(T-t)} N[d_1(S^*)]\} \quad (2.12)$$

$$d_1(S) = \frac{\ln(S/X) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad (2.13)$$

Similarly, for a put option, the approximation is given as

$$P(S_A, T) = \begin{cases} P(S_E, T) + A_1 \left(\frac{S}{S^{**}} \right)^{\gamma_1} & \text{when } S > S^{**} \\ X - S & \text{when } S \leq S^{**} \end{cases} \quad (2.14)$$

where

$$\gamma_1 = \frac{1}{2} \left[-(N-1) - \sqrt{(N-1)^2 + \frac{4M}{K}} \right] \quad (2.15)$$

$$A_1 = \left(\frac{S^*}{\gamma_1} \right) \{1 - e^{q(T-t)} N[-d_1(S^{**})]\} \quad (2.16)$$

In this case S^{**} is the capped price of the stock, below which the option must be exercised, the estimate S^{**} solves the equation

$$g(S^{**}) = X - S^{**} - p(S^{**}) + \left\{ 1 - e^{-q(T-t)} N[-d_1(S^{**})] \right\} \frac{S^{**}}{\gamma_1}.$$

The quadratic approximation approach is suitable for options on stocks paying a continuous dividend, where the dividend yield is constant, such as in currencies and future contracts.

2.3 Numerical Pricing Procedures

Numerical procedures present alternative methods that can be used to value European, American and path-dependent options to which real and accurate analytical solutions are not available. This section examines the three most popular numerical pricing procedures in financial markets; the binomial, finite difference and Monte Carlo methods.

2.3.1 The Binomial Option pricing model

In practice, most traded path-dependent options are discretely sampled, and require closed form solutions to value them. The binomial setup provides an ideal platform for addressing path-dependent option pricing where future probability distributions of the parameter estimates need to be approximated.

The Binomial model is the discrete time equivalent to the geometric Brownian motion process which has been examined at the beginning of this chapter. This model was introduced by Cox, Ross & Rubinstein (1979), on the principle that asset price changes can be decomposed into a discrete sequence of Bernoulli time steps. Applying the Central limit theorem, when the time steps between successive trading intervals approaches zero, the binomial functions are transformed to standard normal functions. Under such circumstances, this model converges rapidly to the continuous framework of the Black-Scholes model.

From its mathematical point of view, the binomial model also assumes that option prices are governed by a random walk $\{S_n, \quad n = 0, 1, \dots, N\}$, describing a sequence of N discrete time-steps that make up a series of paths which define a tree-like structure. The value of an N -step option on a stock is the expected time-discounted value of the future cash flow on an N -step option price path. The model is constructed with the use of the following independently determined parameters;

1. The number N of equal-length intervals (Δt) constituting the entire time to expiry of the option T , such that

$$N\Delta t = T$$

$$k\Delta t = kT/N$$

is the continuous time expression for the discrete time k , where $k = 0, 1, 2, \dots, N$.

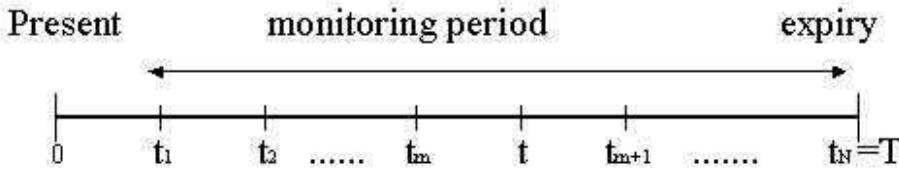


Figure 2.1: Equal length N -discrete time intervals.

2. A continuous-compounded annualized *risk-free rate* r . Since trading takes place in a risk-neutral world, the expected return from an option on a stock is the risk-free interest rate.
3. the volatility per annum σ .

Considering the initial option price S_0 to be non-random, S_k represents the value of the stock at time t_k , the stock thereby assumes discrete values S_k^j , where $j = 1, 2, \dots, m_k$ is the interval value of the stock. Therefore in every successive interval m_{k+1} , the stock will attain a value S_{k+1}^h , $h = 1, 2, \dots, m_{k+1}$ with probability

$$p_{jh}^k = P(S_{k+1} - S_{k+1}^h = S_k^j)$$

where

$$\sum_{h=1}^{m_{k+1}} p_{jh}^k = 1, \quad 0 \leq p_{jh}^k \leq 1$$

For the purpose of this thesis we will assume more general mathematical notations to express the binomial parameters in the Cox, Ross and Rubinstein (CRR) model. In their numerical procedure Cox, Ross & Rubinstein (1979) considered that at each time interval Δt , the price of a risky asset S_0 moves from its value at time $t_0 = 0$ to one of two new values, uS_0 and dS_0 as shown in fig. 2.2

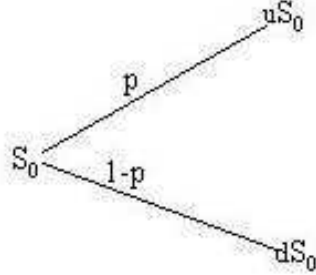


Figure 2.2: Possible movement of stock price at the end of each time interval.

Based on the theory of the discrete price movement discussed above, the following discrete model components can be computed;

1. The up-factor parameter $u = e^{\sigma\sqrt{\Delta t}}$, such that the expected return from the stock price in time Δt is $\mu\Delta t$, with a variance of $\sigma^2\sqrt{\Delta t}$

2. The down-factor parameter $d = 1/u$, This represents the proportional change of a downward movement in the stock price. We recall that under arbitrage-free conditions

$$d < e^{rT} < u \quad (2.17)$$

An important property of the multi-period binomial model illustrated in fig. 2.3 is that as the number of timesteps increases, the length of each time interval $\Delta t \rightarrow 0$, and subsequently, the distribution of $\log S_T/S_0$ tends to that of the normal distribution.

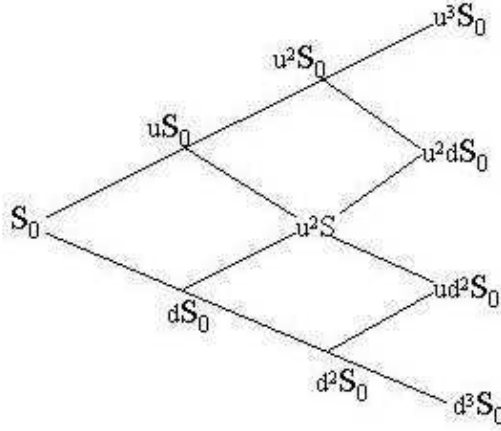


Figure 2.3: A 3-step binomial lattice structure.

Using our notation for u and d above, we will consider that

$$u.d = 1$$

by implication, an asset price that shifts once upwards followed by a downward move in two successive time intervals, will retain its initial value two periods before. This identity describes the theory behind the *recombining binomial tree* in fig. 2.3. Therefore, at discrete time t_k , the underlying asset can assume one of $m_k = k + 1$ possible values

$$S_{k+1} = \begin{cases} u.S_k^h & \text{with probability } p \\ d.S_k^h & \text{with probability } 1 - p \end{cases} \quad (2.18)$$

where the probability of an up-tick to $u.S_k^h$ is expressed as

$$p = \frac{e^{\mu\Delta t} - d}{u - d},$$

and respecting the assumptions of binomial probability, that of a downward move to $d.S_k^h$ is $q = 1 - p$. By implication, taking S_k^h to be the stock price at the beginning of the time interval Δt , the expected value of the stock price at the end of the time interval (S_{k+1}^h) is

$$\mathbf{E}(S_{k+1}^h | S_k^h) = S_k^h e^{r\Delta t} \quad (2.19)$$

which implies that the discounted stock price is a martingale.

At time $2\Delta t$, there exist three possible outcomes of the stock price; $S_0 u^2$, $S_0 u d$ and $S_0 d^2$ (for mathematical simplification, we write $S_0 u = u.S_k^h$, where $k = 0$). We recall here that in the case of lookback options, the stock price at time t_k only depends on the number of upward moves that have occurred. At the end of n subperiods (at time $n\Delta t$), suppose there have been h up-ticks, we obtain $n + 1$ stock prices with the general expression

$$S_k^h = S_0 u^k d^{h-k} \quad k = 0, 1, \dots, n \quad (2.20)$$

which yields the stock price at each node of the binomial tree structure. Equation (2.20) otherwise implies that though there might be several nodes at a time t_k , they actually represent the same stock price S_k^h , this property will be of particular significance in our pricing procedure in the next chapter. The tree describes the entire sequence of random variables $\{S_k^h\}_{k=0}^n$ that define the stock price process. If $f(\cdot)$ is the payoff function for the option at expiry $T = N\Delta t$, we can assign the option price

$$V_k^N = f(S_k^N) = f(S_0 u^k d^{N-k}). \quad (2.21)$$

Let the present value of the option S_k^h at time t_k be denoted by $V_k^h = V(S_k^h, t_k)$. Then the expected value of the option at the end of the time interval t_{k+1} is

$$V_{k+1}^h = e^{-r\Delta t} [pV_{k+2}^{h+1} + (1-p)V_{k+1}^{h+1}] \quad (2.22)$$

The no-arbitrage condition in expression (2.17) guarantees that $0 < p < 1$, therefore equation (2.22) can be considered as an expectation of the form

$$V_k^h = e^{-r\Delta t} \mathbf{E}_p[f(S_k^h)] \quad (2.23)$$

where $f(S_k^h)$ is the payoff function , which implies that the discounted stock price is a martingale.

This special property means that at expiry time $T = t_n$, if V_T is the value of the option, then

$$\mathbf{E}_p[V_T] = S_0 e^{rT}.$$

The above expression simply means that the expected value of the option at some point in future is the same as that of the risk-free asset, in other words, no matter the equivalent amount of money that could be further invested in the risk-free asset, the investment should be expected to grow at the risk-free rate in either cases; this is referred to as the *risk-neutral probability measure*.

The value of a European call and put option at expiry can now be given as

$$V_n^h = \max[S_n^h - K, 0], \quad k = 0, \dots, n \quad (2.24)$$

and

$$V_n^h = \max[K - S_n^h, 0] \quad (2.25)$$

respectively.

Given these expressions, the subsequent option values on the recombining binomial tree; V_k^h , for $h = 0, \dots, k$, and $k = n - 1, n - 2, \dots, 0$, can be calculated using backward recursion. If we assume an N -period model (N -steps), where $T = N\Delta t$, following our binomial theory,

$$V_0 = e^{-rT} \sum_{k=0}^N \binom{N}{k} p^k (1-p)^{N-k} V_k^h \quad (2.26)$$

i.e.

$$V_0 = e^{-rT} \sum_{k=0}^N \binom{N}{k} p^k (1-p)^{N-k} f(S_0 u^k d^{N-k}) \quad (2.27)$$

Therefore, the payoffs at each node in the N -period model can be expressed as functions of the payoffs in an $(N + 1)$ -period model:

$$f(S_0 u^k d^{N-k}) = e^{-r\Delta t} [p f(S_0 u^{k+1} d^{N-k}) + (1-p) f(S_0 u^k d^{N+1-k})] \quad (2.28)$$

This confirms that our formula is valid for N as well as $(N + 1)$ -period models.

Summarily, European and American path-dependent options are evaluated by starting at the expiry time which marks the end of the tree where the value of the option is known, and working backward (referred to as recursion) or by summation. Assuming risk-neutrality, the value of the option at each node at time $T - \Delta t$ is the expected value in time T discounted at rate r for a time period Δt .

2.3.1.2 Recursion

Given the option payoffs at the final nodes V_k^N for all $0 \leq k \leq n$ and the appropriate parameters u, d, p , the previous N values can be calculated by the recursive formula

$$V_k^{n-1} = e^{-r\Delta t} [pV_{k+1}^n + (1-p)V_k^n] \quad \text{for} \quad 0 \leq k \leq n-1 \quad (2.29)$$

Based on the above expression, N operations will be required to obtain the option values for the period $N-1$, and the period $N-2$ will require $N-1$ operations. Therefore to find the present value of the option at $t=0$, an aggregate of

$$N + (N-1) + (N-2) + \dots + 1 = \frac{N(N-1)}{2}$$

operations are required to compute the value of the option, this method therefore takes $O(\frac{1}{2}N^2)$ operations to obtain the option value.

2.3.1.4 Summation method

An alternative to the recursive approach above is through the discrete risk-neutral expectation formula in equation 2.27. In this case, the binomial coefficients are calculated as follows

$$BN_k = \binom{N}{k} p^k (1-p)^{N-k}, \quad 0 \leq k \leq N.$$

summed-up and each of them is further multiplied by the corresponding payoff value. This product is discounted to obtain the present option value

$$V_0 = e^{-rT} \sum_{k=0}^N BN_k f(S_0 u^k d^{N-k})$$

for any valid combinations of u, d , and p .

With any changes in r or σ , the binomial coefficients BN_k or the asset values $S_0 u^k d^{N-k}$ will have to be recomputed. An N -period model will need $N+1$ binomial coefficients

$$BN_k = \binom{N}{k} p^k (1-p)^{N-k} = \binom{N}{k} \frac{1}{2^N}, \quad 0 \leq k \leq N.$$

The actual option value is then obtained as

$$V_0 = \frac{e^{-rT}}{2^N} \sum_{k=0}^N \binom{N}{k} f(S_0 u^k d^{N-k}) \quad (2.30)$$

The advantage in this method is that, during computations, the binomial coefficients BN_k and asset values $S_0 u^k d^{N-k}$ can be calculated and stored in an array, only to be recalled each time the payoff function or the parameters $r, \sigma, \Delta t$, or N change. Normally, this method requires $O(\frac{1}{2}N^2)$ operations to calculate the binomial coefficients. In a case where $r, \sigma, \Delta t$, or N change often, the recursive method will be slightly faster than the summation approach, though the rate of convergence remains the same.

In general, under the appropriate limiting conditions ($N \rightarrow \infty$), the binomial process described above converges to a lognormal price process, and subsequently, the binomial pricing formula converges to the Black–Scholes formula.

2.4 Other Numerical Approaches

In addition to the binomial method, other numerical methods have also been proposed to solve the problems related to pricing American and other path-dependent options where analytical formulas are unavailable.

2.4.1 The Finite Difference Method

The finite difference methods attempt a numerical solution to the Black–Scholes equation by approximating the partial derivatives through either an explicit or implicit solution. In this case the partial differential equations (PDE) are transformed into a set of simple difference equations which are then solved iteratively. A most valuable property of the finite difference method is the fact that it can handle optimal stopping and free boundary problems. In addition, the method is capable of evaluating any type of linear, non-linear as well as Ordinary Differential Equations. To illustrate this method, consider the value of a European put option $V = p(S, t)$ that satisfies the Black–Scholes equation

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (2.31)$$

with boundary conditions $p(S, t) \approx K$ as $S \rightarrow 0$ and $p(S, t) \approx 0$ as $S \rightarrow \infty$, where the final condition

$$p(S, T) = \max\{K - S, 0\}$$

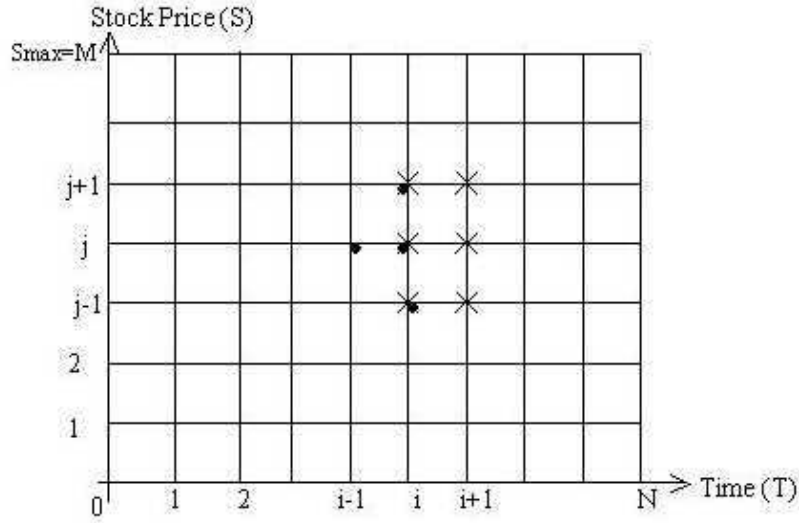


Figure 2.4: Grid Lattice structure for finite difference methods.

Further consider N -subintervals that make up the time space from current time to the maturity of the option $[0, T]$, where $\Delta t = T/N$ as illustrated in fig. 2.4. If we let S_{max} be a sufficiently high stock price such that when it is reached, the put option virtually has no value, then for M sub-intervals making the entire price path of the option;

$$\Delta S = S_{max}/M. \quad (2.32)$$

A solution to equation (2.31) can be obtained by using the interior grid points in fig. (2.4) to approximate each term in the equation. The solution to the partial differential equation is obtained through two finite difference implementations: the explicit method and the implicit method.

2.4.1.1 The explicit finite difference method

This approach calculates the value of the derivative security at time $t + \Delta t$ as a function of the values at time t . Using appropriate boundary conditions, it solves the PDE by proceeding

backward in time through small intervals until the optimal path to every t is found. Let

$$\frac{\partial v}{\partial S} \approx \frac{v\{i\Delta t, (j+1)\Delta S\} - v\{i\Delta t, (j-1)\Delta S\}}{2\Delta S}, \quad (2.33)$$

$$\frac{\partial^2 v}{\partial S^2} \approx \frac{v\{i\Delta t, (j+1)\Delta S\} - v\{i\Delta t, (j-1)\Delta S\} - 2v\{i\Delta t, j\Delta S\}}{\Delta S^2} \quad (2.34)$$

$$\frac{\partial v}{\partial t} \approx \frac{v(i\Delta t, j\Delta S) - v\{(i-1)\Delta t, j\Delta S\}}{\Delta t} \quad (2.35)$$

Further let $S = j\Delta S$, then substituting these three equations into equation 2.31, yields

$$a_j v\{i\Delta t, (j-1)\Delta S\} + b_j v(i\Delta t, j\Delta S) + c_j v\{i\Delta t, (j+1)\Delta S\} \approx v\{(i-1)\Delta t, j\Delta S\} \quad (2.36)$$

where

$$\begin{aligned} a_j &= -\frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t \\ b_j &= 1 - \sigma^2 j^2 \Delta t - r\Delta t \\ c_j &= \frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t \end{aligned}$$

By letting $v_{i,j} \approx v(i\Delta t, j\Delta S)$, equation 2.36 can now be expressed as a finite difference scheme of the form

$$a_j v_{i,j-1} + b_j v_{i,j} + c_j v_{i,j+1} = v_{i-1,j} \quad (2.37)$$

In this case our boundary conditions become $v_{i,0} = K$, $v_{i,M} = 0$, where $i = 0, 1, \dots, N-1$ with the final condition for a European put option as

$$p_{N,j} = \max\{K - j\Delta S, 0\}, \quad j = 0, 1, \dots, M.$$

Equation (2.37) is explicit in the sense that all $v_{i,j}$, for $j = 0, 1, \dots, M$ can be obtained for the calculation of $v_{i-1,j}$. In the absence of these property, an implicit approach is used to solve the partial differential equation.

2.4.1.2 Implicit finite difference methods

The implicit finite difference approximation consist of averaging the forward and backward difference approximations. In contrast to the explicit methods, the The implicit method calculates

the value of the derivative security at time t as a function of values at time $t + \Delta t$. This method therefore solves systems of simultaneous linear equations to develop calculations from time t to $t + \Delta t$.

In solving the PDE implicitly, we make use of the Crank–Nicolson method which approximates the partial derivatives in equation (2.31) at $(i + \frac{1}{2}, j)$ by considering that

$$\frac{\partial v}{\partial S} \approx \frac{1}{2} \left\{ \frac{v[i\Delta t, (j+1)\Delta S] - v[i\Delta t, (j-1)\Delta S]}{2\Delta S} + \frac{v[(i+1)\Delta t, (j+1)\Delta S] - v[(i+1)\Delta t, (j-1)\Delta S]}{2\Delta S} \right\}, \quad (2.38)$$

$$\begin{aligned} \frac{\partial^2 v}{\partial S^2} \approx \frac{1}{2} \left\{ \frac{v[(i+1)\Delta t, (j+1)\Delta S] + v[(i+1)\Delta t, (j-1)\Delta S]}{\Delta S^2} - \frac{2v[(i+1)\Delta t, j\Delta S]}{\Delta S^2} \right. \\ \left. + \frac{v[i\Delta t, (j+1)\Delta S] + v[i\Delta t, (j-1)\Delta S] - 2v(i\Delta t, j\Delta S)}{\Delta S^2} \right\} \end{aligned} \quad (2.39)$$

$$\frac{\partial v}{\partial t} \approx \frac{v[(i+1)\Delta t, j\Delta S] - v(i\Delta t, j\Delta S)}{\Delta t} \quad (2.40)$$

Substituting these three equations into equation (2.31), we obtain

$$\begin{aligned} a_j v[i\Delta t, (j-1)\Delta S] + (1 + b_j) v(i\Delta t, j\Delta S) + c_j v[i\Delta t, (j+1)\Delta S] \\ = -a_j v[(i+1)\Delta t, (j-1)\Delta S] + (1 - b_j) v[(i+1)\Delta t, j\Delta S] \\ - c_j v[(i+1)\Delta t, (j+1)\Delta S]. \end{aligned} \quad (2.41)$$

where

$$\begin{aligned} a_j &= -\frac{1}{4}\sigma^2 j^2 \Delta t + \frac{1}{4} r j \Delta t \\ b_j &= \frac{1}{2} r \Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t \\ c_j &= -\frac{1}{4}\sigma^2 j^2 \Delta t - \frac{1}{4} r j \Delta t \end{aligned}$$

Finally for $v_{i,j} \approx v(i\Delta t, j\Delta S)$, equation (2.41) can then be written as

$$a_j v_{i,j} + (1 - b_j) v_{i,j} + c_j v_{i,j+1} = -a_j v_{i+1,j-1} + (1 - b_j) v_{i+1,j} + c_j v_{i+1,j+1} \quad (2.42)$$

It should be recalled that though finite difference methods can be used to value both European and American style options, they are inappropriate in cases where the option's payoff also depends

on the entire past history of the state variable. The finite difference methods have proven to be more efficient than comparable binomial approximations in cases where a large number of option values are to be calculated.

Although the finite difference approach makes use of well-established mathematical theory which defines stability, convergence and accuracy, experience has shown that they can be hard to implement if there are many variables and/or complex boundary conditions. In such cases, the explicit methods fail to converge.

2.4.2 The Monte Carlo Simulations

The Monte Carlo approach capitalizes on the risk neutrality principle of the Black–Scholes method to simulate sample paths of the underlying state variables over some specific time horizon. The method proceeds by evaluating the discounted cash flows of a security in each sample path, and averaging these cash flows over all sample paths. In other words, it involves the use of sampling in order to evaluate functions of stochastic quantities. For an example, let us consider a lookback call option in which we require the historical maximum over the period $[t_1, T]$ and at times $t_1, t_2, \dots, t_m, t, t_{m+1}, \dots, t_N = T$. The maximum payoff M_t for $t \in [t_m, t_{m+1})$ is given by

$$M_t = \max\{S_{t_1}, S_{t_2}, \dots, S_{t_m}\} \quad \text{with } M_t = 0 \text{ for } t < t_1$$

Following the risk–neutral valuation principle, the option has a call value at $t = 0$ given by

$$C = e^{-rT} E[\max\{0, M_T - K\}]$$

For these type of options, the Monte Carlo simulations methods have proven to be particularly useful in obtaining the final value of the option.

The method computes an estimate of the expected value of the discounted payoffs over the space of sample paths of stock prices or interest rates during simulation. We recall that the solution to the stochastic differential equation for a derivative security is

$$S_t = S_0 e^{\{(r - \frac{\sigma^2}{2})t + \sigma Z_t\}} \quad (2.43)$$

where r is the continuously compounded rate of return, and Z_t is the standard normal random variate. From the above expression, it is clear that the stock price is lognormal and the

continuously compounded return is normal, the expected future value of the option is

$$E[S_t] = S_0 e^{rt}$$

2.4.2.1 Simulation Procedure

To simulate stock prices using the Black–Scholes model;

- Independent replications of the stock price at time $t + \Delta t$ are generated using the risk–neutral measure. For each simulated path, we evaluate the discounted cash flows of the derivative security as determined by the structure of the security in question;

$$S_{t+\Delta t}^i = S_t e^{\{(r - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}Z^{(i)}\}} \quad (2.44)$$

for $i = 1, 2, \dots, n$, where $Z^{(i)} \sim N(0, 1)$,

- Average the n discounted cash flows over the sample paths

$$\hat{S}_t = \frac{1}{N} \sum_{t=1}^N S_t$$

- Compute the standard deviation

$$\hat{\sigma}_{\hat{S}_t} = \sqrt{\frac{1}{N-1} \sum_{t=1}^N (S_t - \hat{S}_t)^2}$$

In general, under the assumption of independence, running $k \times n$ number of simulations will result in a reduction in the estimation variance by $1/k$.

With the availability of powerful computers and easy to use software, Monte Carlo simulations have become increasingly useful in handling most options that previously could not be valued through analytic methods, however the efficiency of these methods has remained a source of worry. One setback is that the space complexity (i.e the memory requirements) in MC simulations is linear in the number of state variables, subsequently, the computation time is exponential($2^T - T$). Secondly the need to increase efficiency necessitates the use of a high number of samples through multiple replications, and thereby significantly greater execution times than those of finite difference methods. The basic fact that the random number generator is not robust, and may yield

a different set of numbers on a new run, variation will make it difficult to determine the exact price of the option. Moreover the early exercise condition in American options cannot be easily addressed through MC simulations.

In view of these prevailing setbacks, a number of steps have been taken to reduce the number of replications while sharpening the inference. In related publications, some authors have proposed variance reduction techniques such as the use of antithetic as well as control variates, Moment Matching methods, Conditional and Quasi-Monte Carlo simulation techniques to reduce the variance arising from the random generation of numbers during simulations.

The main objective of the variance reduction techniques is to reduce the estimation variance without increasing the run time of the Monte Carlo Algorithm. In this thesis we briefly mention two of the most popular techniques implemented in option pricing.

2.4.2.2 The method of Antithetic variates

This method capitalizes on the standard normal distribution property of $Z^{(i)}$ in equation (2.44). It is considered that since $Z^{(i)}$ is standard normally distributed, so too is $-Z^{(i)}$. Since the two paths are negatively correlated to each other, the estimation variance will decline. During the generation of paths, two paths each are generated, a primary path and its antithesis. Therefore two intermediate estimates ; $\hat{\theta}_1(Z^{(i)})$ and $\hat{\theta}_2(Z^{(i)})$ can be constructed by substituting $-Z^{(i)}$ for $Z^{(i)}$ in equation (2.44).

The final estimate is obtained through the average;

$$\hat{\theta}^{AV} = \frac{(\hat{\theta}_1 + \hat{\theta}_2)}{2}$$

By averaging the two estimators, a slightly lower variance is obtained

$$Var[(\hat{\theta}_1 + \hat{\theta}_2)/2] = Var(\hat{\theta}_1)/4 + Var(\hat{\theta}_2)/4 + Cov(\hat{\theta}_1 + \hat{\theta}_2) \quad (2.45)$$

In another approach, Kemma & Vorst (1990) introduced the method of control variates as an approach to variance reduction.

2.4.2.3 The control variates approach

This method replaces the evaluation of an unknown expectation with the evaluation of the difference between the unknown quantity and a related quantity whose expectation is known.

In this case, the known solution denoted "control variate" will be used to impose bias-correction on the simulated estimates. Consider our Monte Carlo estimate to be $\hat{\theta}$, and let its similar, though not exactly the same closed form solution be θ' , where θ' has the exact price from the closed-form equation given as θ'_0 . The direction of the bias from the simulated algorithm can be analyzed by comparing θ' and θ'_0 . This bias is then used to correct the original estimate $\hat{\theta}$ as follows:

$$\hat{\theta}_c = \hat{\theta} + (\theta'_0 - \theta')$$

For a reduction in variance, $Var[\hat{\theta}_c] < Var[\hat{\theta}]$, we can then compute the variance for θ'_0 as follows:

$$\begin{aligned} Var[\hat{\theta}_c] &= Var[\hat{\theta} + (\theta'_0 - \theta')] \\ &= Var[\hat{\theta} - \theta'] \\ &= Var[\hat{\theta}] - Var[\theta'] - 2Cov(\hat{\theta}, \theta') \end{aligned} \tag{2.46}$$

Therefore,

$$Var[\hat{\theta}_c] < Var[\hat{\theta}] \quad \text{if} \quad Cov(\hat{\theta}, \theta') > \frac{1}{2}Var[\theta']$$

We note here that the greater the similarity between the original problem and its control variate, the greater is the benefit from this technique.

Chapter 3

The Frequency Distribution

Approach to Option Pricing

The presence of complex options (especially path-dependent options) in current financial markets has increased the demand for suitable pricing procedures to value them. This chapter fully describes the use of the discrete time frequency distribution approach to obtain exact solutions for both American and European lookback options. Neave & Slavinsky (2001) use the properties of a Gaussian generating function to find Markovian relations among bundles of paths within a binomial lattice structure. This Markovian relation reduces the computational complexity of a European option to cubic polynomials in T (T^3), and quadratic in T (T^4) for the American option. In similar discrete time approaches, Hull (1999) used the distribution of the ratio of maximum stock prices achieved up to time T and a time T stock price to calculate the exact price for European options. Hull & White (1993) also suggested an interpolation method which estimates the time zero value of the American options.

In this chapter we will outline the price process and give a call option defined on it. With the aid of some examples we will proceed to show how the frequency distribution matrix is obtained analytically, inspired by the procedures developed by Neave & Slavinsky (2001)

3.1 The Valuation Algorithm

3.1.1 The Basic Model

The recombining binomial process that will be used in the analysis is generally based on an underlying Bernoulli process $\{X_s\}, s = 1, 2, \dots, T$. To begin, let $\{X_s\}$ be independent, and identically distributed random variables such that X_s assumes the value 1 with probability π , $1 > \pi > 0$, and 0 with probability $1 - \pi$. We further let

$$N_s \equiv \sum_{r=1}^s X_r; \quad s = 1, 2, \dots, T \quad (3.1)$$

and

$$J_s \equiv 2N_s - s; \quad s = 1, 2, \dots, T \quad (3.2)$$

In the context of this study, the J_t will be known as *index variables* and the subsequent values obtained from the J_t will be referred to as node indices. the asset price process $\{S_t\}, t = 1, 2, \dots, T$ is thereby defined by:

$$S_t \equiv u^{J_t} \quad (3.3)$$

where $S_0 = 1$ and $u > 1$.

Moreover, $u = e^{\sigma\sqrt{\Delta t}}$, where σ is the process volatility and Δt is the time interval. In addition $q \equiv (R - u^{-1})/(u - u^{-1})$ is the martingale probability. Finally $R = 1 + r$ is the one period risk free discount rate, throughout this study, this will be assumed to be a constant.

Let p be any realized price path from time 0 to time T , and let $M(p)$ be the maximum price attained along path p . Then the value of the European maximum call option is given as

$$C(K) \equiv E_Q \sum_{p \in P} \{ [M(p) - K]^+ q(p) \} / R^T \quad (3.4)$$

where E_Q means expectation under the martingale, P is the set of all possible paths p described by the process, $q(p)$ is the martingale probability of realizing path p , K is the strike price and $X^+ \equiv \max\{X, 0\}$.

It is important to note here that the lookback options are also defined in terms of the amount by which the maximum stock price achieved during the life of the option exceeds the final stock price.

Equation (3.4) may comprise several paths, for example (p, p') such that $M(p) = M(p')$, enumerating such paths separately may cause computational inefficiency. To avoid this, it is advisable to decompose (3.4) into components determined on a bundle-by-bundle basis:

$$C(k) \equiv \sum_j \sum_m \{a_{T-2j, T-m} [u^{T-m} - K]^+ q^{T-j} (1-q)^j / RT\} \quad (3.5)$$

In equation 3.5 above, j and m both run from 0 to T . By definition, all paths in $b(j, m)$ attain the same maximum and end at the same node. As a result, all paths in $b(j, m)$ contribute the same amount to $C(k)$, therefore it is only necessary to find the numbers of paths a_{jm} for all possible values of j and m .

The next step is to find a frequency distribution $\mathbf{A}(T)$ describing the above events. In the procedure which is described in detail in the next section, a coefficient $a_{jm} = 0$ is assigned to all infeasible combinations of j and m , such that the resultant value is not affected.

3.1.2 Empirical derivation of a frequency distribution matrix

This section gives a detailed description of how the frequency distribution $\mathbf{A}(T)$ is derived for any given value of T . Initially an example will be used, then by straightforward induction from this example, we can derive $\mathbf{A}(T)$ for any arbitrary T .

Consider the 5-period recombining binomial tree such as the one in table 3.1. The origin of the tree is considered to have a node value of 0, while the evolving node values of the tree; for example, 5, 3, 1, -1, -3, -5 at time 5, represent the index of the binomial process.

Using the binomial tree in table 3.1, we can easily derive an example of the frequency distribution matrix $\mathbf{A}(T)$ as shown in 3.2. The corresponding node indices in table 3.2; that is, 5, 3, 1, ..., -5, now become the row headings in $\mathbf{A}(5)$. On the other hand, the column headings represent the distribution of maxima; 5, 4, ..., 0, attained by the paths in the tree. The elements in $\mathbf{A}(5)$ show how many price paths have a maximum m on their way to the node j . By implication, $a_{34} = 1$ means there is 1 path of length 5 ending at the node 3 after having attained a maximum of 4 at some time over the four periods. All the cells with infeasible path combinations are filled with the value 0.

Feasible price indices	Realized price indices					
5						5
4					4	
3				3		3
2			2		2	
1		1		1		1
0	0		0		0	
-1		-1		-1		-1
-2			-2		-2	
-3				-3		-3
-4					-4	
-5						-5
Time	0	1	2	3	4	5

Table 3.1: A 5-step recombining binomial tree.

bf A(4)	Maximum along the path					
Node indices	5	4	3	2	1	0
5	1	0	0	0	0	0
3	0	1	4	0	0	0
1	0	0	1	4	5	0
-1	0	0	0	1	4	5
-3	0	0	0	0	1	4
5	0	0	0	0	0	1

Table 3.2: A 5-period frequency distribution matrix.

3.1.3 Analytic construction of Frequency Distribution Matrices

In our procedure to establish analytic expressions for the frequency distribution matrix $\mathbf{A}(T)$, we let

$$(t, j) \equiv t!/j!(t-j)!$$

Rapid computational explorations will show that for paths of length 5, the distribution matrix $\mathbf{A}(5)$ will take the form in table 3.3, where the row headings are the nodes, $5, 3, \dots, -5$ while the column headings are the path maxima attained; $5, 4, \dots, 0$.

In table 3.3 and 3.4, the differences between binomial coefficients describe the number of paths in the indicated bundles. Following our earlier explanation, cells representing empty bundles bear no

Maxima	5	4	3	2	1	0	
Path ends							Row totals
5	(5,0)						
3		(5,0)	(5,1) -(5,0)				
1			(5,0)	(5,1) -(5,0)	(5,2) -(5,1)		
-1				(5,0)	(5,1) -(5,0)	(5,2) -(5,1)	
-3					(5,0)	(5,1) -(5,0)	
-5						(5,0)	

Table 3.3: Analytic form of $A(5)$.

entries. These empty bundles will be assigned a coefficient of zero in the course of our computations and subsequently do not affect the value of our outcome.

A close assessment of the analytically obtained matrices will show that both acquire the same illustrated form. A detailed algorithm of the procedure for generating the frequency distribution matrices will be as follows;

1. Generate an $(n + 1) \times (n + 1)$ diagonal matrix
2. An $(n + 1)$ column vector of binomial coefficients is generated, and merged from the right with the diagonal matrix to yield a the matrix $\mathbf{A}$ with dimensions $(n + 1) \times (n + 2)$
3. Replace the current element of $\mathbf{A}[n, n + 1]$, which is now 0 by definition, with the element generated as $(2 * \mathbf{A}[n, (n + 2)] - \text{sum}(\text{all elements in the row}))$
4. Substitute $(n + 2)$ of the diagonal elements stemming upward from $\mathbf{A}[n, n + 1]$ with the new element obtained in (3) above.
5. Steps 1 to 4 are repeated for all other $\mathbf{A}[i, n + 1]$ elements with $i \in [n - 1]$

3.2 The case of a European Option

This section presents the sequence of procedures involved in the use of frequency distribution matrices to value European type options

Maxima	8	7	6	5	4	3	2	1	0	
Path Ends										Row Totals
8	(8,0)									
6		(8,0)	(8,1) -(8,0)							
4			(8,0)	(8,1) -(8,0)	(8,2) -(8,1)					
2				(8,0)	(8,1) -(8,0)	(8,2) -(8,1)	(8,3) -(8,2)			
0					(8,0)	(8,1) -(8,0)	(8,2) -(8,1)	(8,3) -(8,2)	(8,4) -(8,3)	
-2						(8,0)	(8,1) -(8,0)	(8,2) -(8,1)	(8,3) -(8,2)	
-4							(8,0)	(8,1) -(8,0)	(8,2) -(8,1)	
-6								(8,0)	(8,1) -(8,0)	
-8									(8,0)	

Table 3.4: Analytic form of $A(8)$.

3.2.1 Time T Payoff Matrix

The matrix $A(T)$ is used to define a matrix of terminal option payoffs $B(T)$ that shows the distribution of option payoffs multiplied by the frequency of their occurrence. In the terminal option payoff $B(T)$ each element takes the form

$$b_{T-2j, T-m} = a_{T-2j, T-m} \cdot (u^{T-m} - K)^+ \quad (3.6)$$

$$j = 0, \dots, T; \quad m = 0, \dots, T$$

where a_{ij} is the frequency obtained from $A(T)$ and K is the option strike price. Thus b_{ij} is the (conditional) call option value for a T -period bundle of price paths that all finish at node i and attain a path maximum j . The probability of realizing any path in $b(T - 2j, T - m)$ is

$$q^{T-j}(1-q)^j$$

3.2.2 Pricing a European Call option

Following our description of $\mathbf{B}(T)$ above, one can determine that the rows of $\mathbf{B}(T)$ represent the paths finishing at the same nodes and thus having equal martingale probabilities. To obtain the value of the call at $t = 0$, the row sums in $\mathbf{B}(T)$ are multiplied by the corresponding probabilities, these products are then summed and discounted.

$$\mathbf{C}_0 = \mathbf{R}^{-T} \left\{ \sum_{j=0}^T \left[\sum_{m=0}^T b_{jm} q^{T-j} (1-q)^j \right] \right\} \quad (3.7)$$

A detailed examination of most of the literature mentioned so far on the valuation of Financial options, does not seem to reveal any information on computing times. Our current approach on valuing European options, particularly with discrete time methods show a great enhancement in the speed of computations than other current valuation methods. Moreover our analytic approaches to the discrete time problem are as easy to compute as continuous time analytic form solutions. It is however important to recall that evaluating techniques using continuous time analytic formulas such as those described by Conze & Viswanathan (1991) may be computationally faster than our discrete time methods.

3.3 The case of an American Option

Our approach to valuing an American option is based on a recursive application of the methods used to value European options. Here we proceed by calculating backward from T , taking one period at a time, we obtain a value for holding the option one more period. We then do a comparison of this value with the value of exercising the option immediately.

The implementation of our method is initially faced with a principal difficulty; that of obtaining a recursive relationship between the current value $\mathbf{A}(t)$ and the period before $\mathbf{A}(t-1)$, $t = 1, 2, \dots, T$ that will facilitate the use of bundling as in the case of the European options. From a mathematical point of view, this approach is necessary as a means to facilitate the programming of calculations.

3.3.1 Recursions

We begin our search for a backward recursion by examining the forward one. Consider our frequency distribution matrix $\mathbf{A}(t)$ and let $a_{j,m}^t$ be an element of $\mathbf{A}(t)$ at time period t . To

ascertain that the paths in the bundle always possess a maximum that is greater or equal to the paths' endpoints, we recall that $m \geq j$, and attains a non-zero value in $\mathbf{A}(t)$. On the other hand, all infeasible combinations where $m < j$ are accordingly assigned the value zero.

If we consider a down movement in which we extend the t -period paths down at time $t - 1$, we obtain a resulting matrix of bundles which we denote here as

$$\mathbf{A}(t + 1, d).$$

With reference to an element $a_{j,m}^t$ of $\mathbf{A}(t)$ at time t , the corresponding entry for a downward movement in $\mathbf{A}(t + 1, d)$ is given as

$$a_{j-1,m}^{t+1}$$

In a similar form of expression for the up movement, the corresponding entry in the matrix of bundles $\mathbf{A}(t + 1, u)$ is given as

$$a_{j+1,max(j+1,m)}^{t+1}$$

It is worthwhile to mention here that an increase in the bundles' path maximum changes the value from $j = m$ to $j + 1 = m$. The resulting frequency matrix at the end of the above calculations is now given as

$$\mathbf{A}(t + 1) = \mathbf{A}(t + 1, u) + \mathbf{A}(t + 1, d)$$

3.3.2 Pricing an American Call Option

To obtain the immediate exercise value of an American option, we proceed by deriving the matrix of terminal option payoffs $\mathbf{B}(t)$ from the frequency distribution matrix $\mathbf{A}(t)$. The next step is to generate the value of immediate exercise for each possible price realization at each stopping time $t = 1, \dots, T$. Using a recursion from $\mathbf{B}(t)$ to $\mathbf{B}(t - 1)$, $t = 1, \dots, T - 1$, we calculate the expected value of continuing. Basically the structure of this recursion is the same as that for the recursion in $\mathbf{A}(t)$. In order to obtain the correct number of paths to take backwards in each part of the expected value calculation, care should be taken to separately record the ups and downs within the recursion. We define

$$C_T(j, m) = b_{T-2j, T-m} q^{T-j} (1-q)^j$$

$$C_T = \sum_{j=0}^T \sum_{m=0}^T C_T(j, m) \quad (3.8)$$

Following the above definition of the option payoff matrix, the recursion for the call values is then expressed as

$$C_{T-1}(j, m) = \max \left\{ b_{T-2j-1, T-1-m}^{T-1} q^{T-1-j} (1-q)^j, \right.$$

$$\left. R^{-1} \left[b_{T-2j, \max(T-2j, T-m-1)}^{T,u} q^{T-j} (1-q)^j + b_{T-2j-2, T-m-1}^{T,d} q^{T-j-1} (1-q)^{j+1} \right] \right\}$$

in the above expressions, $b_{jm}^{T,u}$ is the result from an upward movement in time T of b_{jm}^{T-1} , and $b_{jm}^{T,d}$ is the corresponding result for a downward movement in time T of b_{jm}^{T-1} .

This same procedure is repeated backwards in time for $T-2, T-3, \dots, 0$. The value of the American call is then obtained by comparing the value of immediate exercise with the expected value of continuing to hold the option.

3.4 Computational Results

The basic advancement in path-dependent option pricing proposed by the approach in this thesis compared to alternative option pricing algorithms is the speed of computation while maintaining pricing accuracy. The computational times for several American and European options (where $S = 10$, $n = 40$, $\sigma = 30\%$, $r = .04$ and $X = 10$) are displayed in table 3.5 for some chosen time intervals on a standard IBM PC with $2 * 128MB$ SDRAM and 800 MHz Pentium III processor. These results show a great improvement in the accuracy as well as the computational speed with which the prices of options are obtained.

Due to the relatively straightforward calculation of the values for European options, the valuation times are almost immediate. A noticeable valuation time for European options is first observed for 90 time intervals with 2 seconds. The algorithm requires less than 6 seconds to obtain an exact value for a European Call with 150 time intervals. A Monte Carlo Simulation algorithm (with control variates) from Charnes's *EuroCall.xls* (See Charnes, 2000) using the same parameter values requires 25 minutes with 10,000 replications to obtain the option value

Time intervals	Computational time in seconds		
	European (t_E)	American (t_A)	difference ($t_A - t_E$)
10	0	1	1
20	0	4	3
30	0	9	7
40	0	17	17
50	0	34	34
60	0	53	53
70	1	98	97
80	1	167	166
90	2	251	249
100	2	385	383
110	3	492	489
120	4	583	579
140	5	646	641
150	5	801	796
160	6	998	993

Table 3.5: Computational times for European and American options.

The values of some European and American call options are displayed in table 3.6.

Time steps(N)	European	American
10	1,3475	1,5193
20	1,444	1,628
30	1,4836	1,6727
35	1,5047	1,6966
40	1,5192	1,7129
50	1,5338	1,7294
75	1,5675	1,7673
100	1,5863	1,7886

Table 3.6: Values of European and American maximum options at varying time steps; $S = 10$, $X = 13$, $\sigma = 0.03$, $N = \dots$, $T = 1.5$, $r = 0.08$).

Looking at the results, we notice that an American call on the maximum yields a slightly higher value than its corresponding European counterpart. This gives the relative advantage of the frequency distribution method over other normal pricing algorithms; its ability to handle the free boundary condition in American options, and yield accurate results.

The European and American options priced on our discrete-time framework converge to their

continuous time limits as would be expected (see chapter 2, section 2.1). From our table, we observe that convergence occurs at about 150-periods with a volatility of 0.8. In general it has been realised that the maximum of all highs observed within a single day on a continuous basis can only be higher or equal to the maximum of the option observed just once on that very day. By implication, a pricing algorithm on a continuous framework is relatively better than a discrete pricing algorithm (see Conze and Viswanathan, 1991).

In actual fact, each pricing method has its own characteristics, and identifying the most efficient pricing method depends much more on the option that is to be valued. For cases with few state variables, finite difference and lattice methods yield faster computational time compared to simulation techniques. However, with an increase in the dimensionality of the problem, simulation methods become more efficient and effective, nonetheless at the cost of computational time. The frequency distribution approach discussed in this thesis provides a suitable compromise for free boundary problems, optimal stopping time problems, efficiency and computational time.

Chapter 4

Conclusion

4.0.1 summary

Though the main objective of this thesis was to implement the frequency distribution approach in valuing maximum options, a number of related pricing procedures were reviewed. The ever increasing complexity of derivative securities in the financial markets continuous to create more challenges to accurate option pricing. The choice of which pricing method to implement in pricing an option depends much more on the intrinsic characteristics of the option in question. More simpler path-independent options can easily be priced through the basic Black-Scholes pricing methods, meanwhile real option pricing problems which are often more "exotic" than financial derivatives pricing problems (with several state variables and multi-factor models) frequently rely on numerical techniques.

Because of their characteristic nature, path-dependent options are particularly difficult to price, and usually recommend the use of robust valuation techniques. In the course of this study, we revisited three major approaches popularly used in financial markets to value maximum (path-dependent) options; The finite Difference method, The Binomial lattice approach and the Monte Carlo Simulation techniques.

Given a suitable model, the frequency distribution method provides incomparable computational speeds in pricing discrete time path-dependent options. While Monte Carlo simulation approaches (see J.M. Charnes, 2000) and other discrete time methods described in Hull (1999) may present

almost accurate results, this can only be done at the cost of computational speed. An added advantage of the frequency distribution approach is its ability to yield exact option values without compromising computational speed. Most other comparable methods can only provide approximate values whose accuracy will further depend on error correction techniques.

Most of the examples used to demonstrate our pricing method are based on calls on European and American maximum options, however, the algorithm can also be used to price all other forms of path-dependent Call and put options by simply redefining the payoff function C_t or P_t with respect to the option to be valued.

Mindful of the variety of volatility and convergence issues to be considered in relation to maximum option valuation, this aspect of the analysis has not been duely treated in this thesis. Though convergence of discrete option pricing models to continuous frameworks pose no real difficulty to accurate derivatives pricing, it would be worthwhile to reconsider volatility and convergence properties relating to the approaches presented in this thesis.

Another important property of maximum options that has not been covered by our frequency distribution method presented in this thesis is the issue of jump-diffusion processes in interest rates, which produce considerably high pricing errors for options that mature earlier. It would be of interest to consider an adjustment for jump-diffusion processes in our related model.

Bibliography

- Barone-Adesi, G. & Whaley, R., (1987), Efficient Analytic Approximation of American option Values, *Journal of Finance*, **42**: 301–320.
- Benninga, S., Steinmetz, R. & Stroughair, J., (1993) Implementing Numerical Option Pricing Models, *The Mathematica Journal* **3, 4**: 66–73.
- Betsimas, Dimitris, Kogan, L. & Lo, A., (2000), When is time continuous? *Journal of Financial Economics* **55**: 33–42.
- Black, F. & Scholes, M., (1973), The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* **81** (1973),: 637–634.
- Brennan, M. & Schwartz, E., (1978), Finite Difference Methods and Jump Processes arising in the pricing of contingent claims, *Journal of Finance and Quantitative Analysis* **13**: 462–474.
- Chalasani, P., (1998), Programming Project: Binomial–Model option valuation, *working paper*, University of California.
- Charnes, J., (2000), Using Simulation for Option Pricing, *Proceedings of the 2000 Winter Simulation Conference*, December 2000, Orlando, USA.
- Clelow, L. & Claverhill, A., (1998), On the Simulation of contingent claims, *Journal of Derivatives* **2** (1994),: 66.
- Clelow, Les & Strickland, Chris, (1998), Implementing Derivatives Models, John Wiley & Sons.
- Conze, A. & Viswanathan, A., (1991), Path Dependent Options: The Case of Lookback Options, *Journal of Finance* **XLVI**, August 1991.

- Cox, J. C., Ross, S. A. & Rubinstein, M. (1979), Option Pricing: a Simplified Approach, *Journal of Financial Economics* **7**, (1979),: 229-263.
- Duffie, D., (1996), Dynamic Asset Pricing Theory, 2nd ed., *Princeton University Press*, Princeton, NJ.
- Franke, J., Härdle, W. & Hafner, C., (2001). Einführung in die Statistik der Finanzmärkte, *Springer Verlag*, Berlin.
- Green, C. & Figlewski, S. (1999), Market Risk and Model Risk for a Financial, John Wiley & Sons.
- Härdle, W., (2001), Statistik der Finanzmärkte, *Lecture Notes, e-books*, ISE, Humboldt Universität zu Berlin, Berlin.
- Ho, T. S., (1992). Managing Illiquid Bonds and the Linear Path Space, *Journal of Fixed Income*: 80-94.
- Hull, J. C. (1999), Options, futures and other derivatives, 4th edition, Prentice Hall.
- Jarrow, J. C. & White, A., (1993), Efficient Procedures for Valuing European and American path-dependent options, *Journal of Derivatives*, **1**, 1993: 21-31.
- Jarrow, R. & Turnbull, S., (1970), Derivative Securities, *Southwestern Publishing*.
- Jarion, P., (1988), On jump diffusion processes in Foreign Exchange and Stock Markets, *Review of Financial Studies* **1**,: 427-445.
- Kemma, A. G. & Vorst, A. C. (1990), A pricing method for options based on the average asset values, *Journal of Banking and Finance*, **14**,: 926-938.
- Longstaff, F. A. & Schwartz, E. S., (1998), Valuing American Options by Simulations: A simple Least-Square Approach, *UCLA Working Paper*, (August 1998).
- Neave, E. H. & Slavinsky, S., (2001), A Frequency Distribution Approach to Valuing Maximum Options, *Journal of Derivatives*, 2001
- Neave, E. H., (1997), A Frequency Distribution Method for valuing Average Options, *Astin Bulletin* **27**, 1997: 173-205.

Neave, E. H. & Stein, C. L., (1998), Approximate and Exact Valuation of European Arithmetic Average Fixed Strike Calls, *Eighth Annual Derivative Securities Conference*, Boston.

Neave, E. H. & Ye, G., (1999) Valuation of Arithmetic Average Calls: Further Results, *Proceedings, 9th International AFIR Colloquium*, Tokyo 349–374.

Pliska, S., (1997), Introduction to Mathematical Finance: Discrete Time Models, Blackwell.

Ross, S. M., (1998), An Introduction to Mathematical Finance, *Cambridge University Press*, Cambridge.

Tilley, J. A. (1993), Valuing American Options in a Path Simulation Model, *3th International AFIR Colloquium*, Rome, 249-266.

Wilmott, P., DeWynne, J. & Howison, S., (1993), Option Pricing: Mathematical models and computation, Oxford Financial Press.

Wilmott, P., DeWynne, J. & Howison, S., (1998), The Mathematics of financial engineering, Cambridge University Press.

Description of program Functions

.0.2 XploRe Quantlets

The following functions are components of the the computational program used to illustrate the use of the frequency distribution approach in pricing maximum options, this program can be implemented on an XploRe Interactive Statistical Computing Software.

1. CallOnMax(EuroAm, S, X, sigma, n, time, r)

This program computes European and American maximum call option prices using the quantlets listed below.

- **BinCoeff(t, i)**

Calculates the binomial coefficients required for the construction of the frequency distribution matrix $\mathbf{A}^{(T)}$.

- **freqmat(n)**

Constructs the frequency distribution matrix based on the binomial coefficients calculated in **BinCoeff(t, i)** described above.

- **Pmatrix(S, X, dt, u, n)**

This procedure builds the matrices of option prices at each time period t from the binomial coefficients.

- **freqmatdown(n)**

Constructs the down-split frequency distribution matrices from the frequency matrix $\mathbf{A}(T)$.

- **freqmatup(n)**

Builds up-split frequency distribution matrices from the frequency matrix $\mathbf{A}(T)$.

- **Cverted(Opvalue, i)**

converts a matrix of option prices multiplied by the corresponding frequencies into the option payoff matrix.

- **freqpat(x)**

This procedure creates a matrix containing only 1 and 0 indicating the "pattern" of the frequency distribution.

- **Cvertup(Opvalue, i)**

Uses the "pattern" of the frequency distribution to build an up-split option payoff matrix from the up-split frequency distribution matrix.

- **Cvertdown(Opvalue, i)**

Uses the "pattern" of the frequency distribution to build a down-split option payoff matrix from the up-split frequency distribution matrix.

- **Disc(Df, p, Opvalue, i)**

this procedure calculates the one period discount (for the American options).

Appendix A

The XploRe sample program code

```
; FREQUENCY DISTRIBUTION APPROACH TO VALUING MAXIMUM OPTIONS
proc(ResOut) = CallMaxOp(EuroAm, S, X, sigma, n ,time, r) ;
-----

; Library ;
-----

; See_also pmax, Bincoeff, freqmat, Pmatrix, freqmatdown, ;
freqmatup, Cverted, freqpat, Cvertup, ; Cvertdown,
Disc ;
-----

; Macro      CallMaxOp ;
-----

; Description A procedure that calculates the value of European ;
and American maximum call options ;
-----

; Usage      CallMaxOp(EuroAm,S,X,sigma,n,time,r) ; Input ;
Parameter   EuroAm ; Definition   The type of option to be traded;
;           "EU" for a European option, or ;           "AM"
for an American option. ; ; Parameter   S ; Definition   Price of
```

```

the underlying asset ; ; Parameter  X ; Definition  The exercise
price ; ; Parameter  sigma ; Definition  Volatility of the
relative price change of ;           the underlying asset. ; ;
Parameter  n ; Definition  Number of discrete time periods per
year ; ; Parameter  time ; Definition  Time to expiration of a
call option (in years) ; ; Parameter  r ; Definition  The
risk-free interest rate ; ; Output ; Parameter  ResOut ;
Definition  Gives the value of the call on the maximum option. ;

```

```

-----
; Notes ;

```

```

-----
; Example: CallMaxOp("EU",10,13,0.3,50,1.5,0.08) ;

```

```

-----
; Result: ;           "The value of the European option(EU) is: " ;
Contents of ResOut ;           [1,]    1.5338 ;

```

```

-----
; Keywords ;

```

```

-----
; Reference ;

```

```

-----
; Link ;

```

```

-----
; Author      Lambert A. Ashu
              23122001

```

```

-----
//*****
//This program computes European and American
// maximum call option prices
;

```

```

//*****

//PART I- PRECOMPUTING CONSTANTS

//*****

dt = time/n

u = exp(sigma*sqrt(dt))

d = 1/u

a = exp(r*dt)

p = (a-d)/(u-d)

Df = exp(-r*dt) // use continuous compounding

//*****

//Calculate European option price

//*****

if (EuroAm == "EU")

ncols = cols(Pmatrix(S,X,dt,u,n))

rowSums = Pmatrix(S,X,dt,u,n) * matrix(ncols,1)

i=0

Summ = 0

while (i<ncols)

  Bn = (p^(ncols-1-i)) * ((1-p)^i)

  Summ = Summ + (rowSums[i+1,1] * Bn)

  i = i + 1

endo

"The value of the European option(EU) is: "

ResOut = (Df^(ncols-1)) * Summ

endif

//*****

//Calculate American option price

//*****

```

```

if (EuroAm == "AM")
//initialize option values at maturity
    Opvalue = Pmatrix(S,X,dt,u,n)
//calculate option prices at each time interval using backward recursion
    i=n
    while (i>=1)
        Sec = pmax(Cverted(Pmatrix(S,X,dt,u,(i-1)),i-1), Disc(Df,p,Opvalue,i))
        Opvalue = freqmat(i-1) .* Sec
        i = i - 1
    endo
    "The value of the American option(AM) is: "
    ResOut = Opvalue
endif
//*****
// Display European or American option price at time 0
//*****
endp

//*****
;procedure that selects the maximum element within the
; rows and columns of two matrices
//*****

proc(retval)=pmax(x,y)

    retval = matrix(rows(x), cols(x)) ; an n x m matrix
//x and y are n x m matrices
    i=1
    while (i<=cols(x))

```

```

    j=1
    while (j<=rows(x))
        retval[j,i] = max(x[j,i] | y[j,i])
        j = j + 1
    endo
    i = i + 1
endp

//

endp

//*****
//calculate binomial coefficients
//*****

proc(bm) = BinCoeff(t,i)

    bm=factorial(t)/(factorial(i)*factorial(t-i))

endp

//

//*****
//Building the frequency distribution matrix
//*****

proc(fm) = freqmat(n)

    if (n<=1)
        a = unit(n+1) // creates a unit vector
    endif

```

```

nrow=n+1
ncol=n+2
if (n>1)
  a = 0 .*matrix(nrow, ncol)

  iCounter=-1
  while (iCounter<n)
    iCounter=iCounter+1
    a[iCounter+1,n+2]=BinCoeff(n,iCounter)
  endo
  a[:,1:(n+1)]=unit(n+1) //creates a unit diagonal in matrix a

  j=1
  while (j<floor(n/2+1))
    j=j+1
    k = n-j+2
    i=j
  //
    while (i<=(n-(j-2)))
      a[i,i+(j-1)] = 2*a[k,n+2]-sum(a[k:],2)
      i=i+1
    endo
  //
  endo

endif
fm = a[:,1:n+1] // drop the (n+2)th column.
endp

```

```

//*****
//Building the matrices of option prices at each time period
//*****

proc(pm) = Pmatrix(S,X,dt,u,n)

//create a 1 x n+1 vector of 0's
jvec = 0 * matrix(1, n+1)

jcount = 1
//select maximum between option value and 0
while (jcount<=n+1)
    jvec[1, jcount] = max((S*u^(n+1-jcount)-X) | 0)
    jcount = jcount + 1
endo

mat = 0 * matrix(n+1, n+1)
k = 1
while (k<=n+1)
    mat[k, 1:(n+1)] = jvec
    k = k + 1
endo

pm = freqmat(n) .* mat

endp

//*****

```

```

//Building the down-split frequency distribution matrices
//*****

proc(fmd) = freqmatdown(n)

a = 0.*matrix(n+1, n+1) // a is an n+1 x n+1 matrix of 0's
a[2:(n+1),2:(n+1)] = freqmat(n-1)
//
fmd = a

endp

//*****
//Building up-split frequency distribution matrices
//*****

proc(fmu) = freqmatup(n)

fmu = (freqmat(n))-freqmatdown(n)
endp

//*****
//converting a matrix of option prices multiplied by the
//corresponding frequencies into the option payoff matrix
//*****

proc(x) = Cverted(Opvalue,i)

x = Opvalue / freqmat(i)

```

```

x = replace(x, NaN, 0)

endp

//*****
//creating a matrix coontaining only 1 and 0 indicating the
//"pattern" of the frequency distribution
//*****

proc(fp) = freqpat(x)

fp = x ./ x      // y becomes a unit matrix
fp = replace(fp, NaN, 0)

endp

//*****
//building an up-split option payoff matrix
//*****

proc(x) = Cvertup(Opvalue,i)

fqu = freqmatup(i)
//
x = Cverted(Opvalue,i) .* freqpat(fqu) //building initial matrix X

j = 1
while (j<=ceil(i/2))
    x[j, j*2] = x[j, (j*2)-1]

```

```

        j=j+1
    endo
    x[1,1] = 0

endp

//*****
//Building down-split frequency distribution matrices
//*****
proc(x) = Cvertdown(Opvalue,i)

    x = Cverted(Opvalue,i) .* freqpat(freqmatdown(i))
endp

//*****
//Doing the one period discount (for American options)
//*****
proc(C) = Disc(Df,p,Opvalue,i)

//upward movement
    C1 = p .* Cvertup(Opvalue,i)[1:i, 2:(i+1)]
//downward movement
    C2 = (1-p) .* Cvertdown(Opvalue,i)[2:(i+1), 2:(i+1)]

    C = C1 + C2
endp

/* Input syntax to calculate European and American Calls on the maximum */

```

```
library("xplore")
```

```
CallMaxOp(EuroAm,S,X,sigma,n,time,r)
```

```
/* END */
```